ESSENTIAL FREE DECOMPOSITIONS OF KNOT EXTERIORS

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Abstract. The essential free decomposition conjecture states that every non-trivial knot manifold M contains an essential properly embedded surface which splits M into two handlebodies. It is shown that for any closed connected orientable 3-manifold M, there exists a knot K in M such that $M - intN(K)$ has such decomposition.

1. Introduction

When we wish to accomplish something, it is frequently effective to decompose it into more simple parts, to settle each of parts and to finish the whole. In our 3-manifold theory, there are sevaral decompositions of a 3-manifold. The decomposition into prime manifolds, the torus decomposition, the hierarchy, the Heegaard splitting and et cetera.

In this paper, we deal with a decomposition of the exterior of a knot, like an *essential* Heegaard splitting. Let us start on the following conjecture.

Conjecture A (Neuwirth, 1964) *The fundamental group of the exterior of a non-trivial knot is a non-trivial free product with amalgamation, and the amalgamating subgroup is free.*

This was solved by;

Theorem 1.1 (Culler and Shalen, 1984) *The exterior of a non-trivial knot contains an essential separating properly embedded surface which has non-empty boundary.*

But Neuwirth's geometric conjecture, which is the source of Conjecture A, does not have been solved.

Conjecture B (Neuwirth, 1964) *Let K be a non-trivial knot in S^3. Then there exists a closed surface F in S^3 satisfying the following conditions;*
(1) $F \supset K$,
(2) $F \cap E(K)$ is connected,
(3) $F \cap E(K)$ is essential in $E(K)$.

For prime alternating knots, this was solved by Aumann in 1956. In fact, for prime alternating knots, Aumann proved the following stronger conjecture.

Conjecture C *Conjecture B holds even F is a Heegaard surface of S^3.*

We will expand the Aumann's result in Section 3.

Now we state the essential free decomposition conjecture.

Conjecture EFD (Essential Free Decomposition conjecture) *Let K be a non-trivial knot in S^3. Then the exterior $E(K)$ of K contains an essential properly embedded surface which separates $E(K)$ into two components E_1 and E_2, and the closures of both E_1 and E_2 are handlebodies.*

The algebraic conjecture which relates the EFD-conjecture is;

Conjecture A′ *The fundamental group of the exterior of a non-trivial knot is a non-trivial free product of two free groups with amalgamation, and the amalgamating subgroup is free.*

2. Essential free decomposition

Let M be a compact connected irreducible orientable 3-manifold with non-empty boundary, and F a properly embedded surface with non-empty boundary in M. Then F is said to be *essential* in M if it is incompressible and boundary-incompressible in M. We say that F is *free* in M if $cl(M - N(F))$ consists of handlebodies. Now we say that M has an *essential free decomposition* $(F; V_1, V_2)$ if F is essential, free, separating and two-sided surface in M, $cl(M - N(F)) = V_1 \cup V_2$ and both V_1 and V_2 are handlebodies.

The following Theorem gives a necessary condition for F being free.

Theorem 2.1 (Przytycki, 1983) *Let F be an essential surface properly embedded in the exterior $E(K)$ of a knot K. If $E(K)$ does not contain any closed essential surface, then F is free.*

By combining Theorem 1.1 and Theorem 2.1, we can obtain;

Theorem 2.2 *If the exterior of a non-trivial knot K does not contain any closed essential surface, then K satisfies the EFD-conjecture.*

Concerning Dehn surgery, we have;

Theorem 2.3 *Let K be a knot in S^3. Suppose that $E(K)$ has an essential free decomposition $(F; V_1, V_2)$ and F has non-meridional boundary. If*
(1) F is non-planar
(2) F is planar,
then
(1) $M(K; \partial F)$ is irreducible and $\hat{F}$ is incompressible in $M(K; \partial F)$
(2) $\hat{F}$ decomposes $M(K; \partial F)$ as a connected sum of two lens spaces.
Where $M(K; \partial F)$ denotes the 3-manifold obtained from S^3 by Dehn surgery on K along the boundary slope of F, and $\hat{F}$ is a natural extension of F to $M(K; \partial F)$.

Proof. This follows easily Proposition 2.2.1 and Proposition 2.3.1 in [3]. □

3. Partial settlement of the conjecture C

Let K be a knot and F a closed surface in S^3. Then K is called *alternating with respect to F* if it can be isotoped into a neighborhood $F \times [-1, 1]$ of F so that it has an alternating diagram $\pi(K)$ on F with respect to a projection $\pi : F \times [-1, 1] \to F$.

Theorem 3.1 *Let K be an alternating knot with respect to F. Suppose that K has an alternating diagram $\pi(K)$ on F which satisfies the following properties;*

(1)$F-\pi(K)$ consists of open disks.

(2)Any simple closed curve in F intersecting $\pi(K)$ in two non-double points bounds a

disk in F which intersects $\pi(K)$ in an arc.

Then K satisfies the conjecture B.

Moreover if F is a Heegaard surface of S^3, then K satisfies the conjecture C.

Corollary 3.2 *Let M be a closed connected orientable 3-manifold. Then there exists a knot K in M such that $M - intN(K)$ has an essential free decomposition.*

Corollary 3.3 *Let M be a closed connected orientable 3-manifold. If $H_1(M)=0$, then there exist a knot K in M and a slope γ in $\partial E(K)$ such that $M(K;\gamma)$ is a Haken manifold.*

References

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